Periodic solutions for a $p$-Laplacian-like NFDE system

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Received 28 August 2009; received in revised form 30 October 2010; accepted 22 March 2011
Available online 8 April 2011

Abstract

We consider an $n$-dimensional $p$-Laplacian-like neutral functional differential equation (NFDE) in the form

$$
\frac{d}{dt} \phi_p [x(t) - c(t)x(t-\tau)] + \frac{d}{dt} \nabla F(x(t-\tau)) + \beta(t) \nabla G(x(t-\delta(t))) = e(t),
$$

where $c(t)$ and $\beta(t)$ are sign-changeable.

Using Mawhin’s continuation theorem, we establish some criteria to guarantee the existence of periodic solutions for the above system, which generalize and improve on the corresponding results in related literature.

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1. Introduction

The purpose of this paper is to develop a machinery, which will be used to guarantee the existence of periodic solutions for a $p$-Laplacian-like neutral functional differential equation (NFDE) system as follows:

$$
\frac{d}{dt} \phi_p [x(t) - c(t)x(t-\tau)] + \frac{d}{dt} \nabla F(x(t-\tau)) + \beta(t) \nabla G(x(t-\delta(t))) = e(t),
$$

where $p \in ]1, +\infty[,$ $\phi_p : \mathbb{R}^m \to \mathbb{R}^m$ is given by $\phi_p (u) = (|u_k|^{p-2}u_k)_{k=1}^N$ for all $u = (u_k)_{k=1}^N \in \mathbb{R}^N,$ $\phi_p (0) = 0,$ $F \in C^2(\mathbb{R}^N, \mathbb{R}^N),$ $G \in C^1(\mathbb{R}^N, \mathbb{R}^N), c, \beta, e \in C(\mathbb{R}, \mathbb{R}), \delta \in C^1(\mathbb{R}, \mathbb{R})$, with $c(t + T) \equiv c(t), \beta(t + T) \equiv \beta(t), \int_0^T \beta(s)ds \neq 0$ (This paper will deal with the case $\int_0^T \beta(s)ds > 0,$ and the

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other case \( \int_0^T \beta(s) \, ds < 0 \) can be coped with in the same way.), \( e(t + T) \equiv e(t), \int_0^T e(s) \, ds = 0, \delta(t + T) \equiv \delta(t), \delta'(t) < 1, \forall t \in \mathbb{R}, T, \tau \in \mathbb{R} \) are constants with \( T > 0 \).

In recent years, the functional differential equation (FDE) systems involving \( p \)-Laplacian-like operator come into play in many practical situations frequently, such as, the NFDE systems are often utilized to describe the electrical mechanical phenomena of networks containing lossless transmission. Moreover, the special case of system (1.1) can also be applied to the dynamics of fluids. For example, Euler’s equation governing the flow of an ideal fluid in a conservative force field has the form

\[
a = -(1/\rho) \nabla P + \nabla U,
\]

where \( a \) is the acceleration of fluid particles, \( \rho \) the density, \( P \) the pressure, and \( U \) the potential of mass forces.

In addition, periodic problems involving \( p \)-Laplacian-like operator and gradient-like systems have attracted a plenty of attention with respect to the existence of solution. See [2–14,17,19–27] and [5,6,15,16,21], respectively, and the references therein.

Of the aforementioned works, we mainly mention Jin and Lu in [2] studied the existence of periodic solutions to the third-order \( p \)-Laplacian-like differential equation as follows:

\[
\frac{d}{dt} \phi_p[x''(t)] + f(t, x'(t), x''(t)) + g(t, x(t-\tau(t))) = e(t),
\]

under the cases

\[
\begin{align*}
g(t, x) &\geq -D, & x \leq -d, \\
g(t, x) &\leq D, & x \geq d, & \text{respectively,} \\
g(t, x) &\leq \rho|x|^s + r, & x \leq -d,
\end{align*}
\]

where \( d, D > 0, s > 1, \rho, r \geq 0 \) are constants.

Jin and Lu in [3] continued to study the existence of periodic solutions to the following third-order differential equation with \( p \)-Laplacian-like operator at resonance:

\[
\begin{cases}
\frac{d}{dt} \phi_p[x''(t)] = f(t, x(t), x'(t), x''(t)), & 0 < t < 1, \\
x(0) = 0, & x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i),
\end{cases}
\quad x''(0) = 0.
\]

The speciality of the paper [3] lies in various assumptions on degree of the power with respect to the variable \( z \) in the function \( f(t, x, y, z) \).

Wang and Zhu in [4] investigated a kind of fourth-order \( p \)-Laplacian-like NFDE with a deviating argument in the form

\[
\frac{d^2}{dt^2} \phi_p[(x(t) - \alpha x(t - \delta))^\alpha] = f(x(t))x'(t) + g(t, x(t-\tau(t, |x|_\infty))) + e(t).
\]

The results obtained in [4] indicates that the existence of periodic solutions to the above system is not only related to the deviating argument \( \tau \) but also associated to the delay \( \delta \). In addition, the growth degree with respect to the variable \( x \) in the function \( g(t, x) \) is admitted to be greater than \( p-1 \).
Liu and Yu in [5] considered the periodic solutions for BVPs with \( p \)-Laplacian-like operator in the following form:

\[
\begin{aligned}
\frac{d}{dt} \varphi_p[(x(t) - Cx(t-\tau))] &= g(x(t-\mu)) + e(t), \\
\varphi_p'(x(t)) &= f(x(t)) + \gamma(x(t)) = e(t),
\end{aligned}
\]

Some sufficient conditions to guarantee the existence of periodic solutions for the referred system are established with no restriction being imposed on the damping forces \( (d/dt)\varphi_p F(x) \).

Amster et al. in [6] discussed the periodic solutions for \( p \)-Laplacian-like operator system of the form

\[
\frac{d}{dt} \varphi_p[(x(t) - Cx(t-\tau))] = g(x(t-\mu)) + e(t).
\]

The authors proved the existence of periodic solutions for the above-mentioned system containing a fixed delay by applying the Leray–Schauder degree theory.

Lu in [13] investigated the periodic solutions of a NFDE system in the form

\[
\frac{d}{dt} \varphi_p[(x(t) - Cx(t-\tau))] = g(x(t-\mu)) + e(t).
\]

Some interesting results on the existence of periodic solutions related to the deviating arguments \( \tau \) and \( \mu \) are obtained.

Recently, Du et al. in [17] coped with the periodic solutions of a second-order NFDE system

\[
(x(t) - c(t)x(t-\tau))^\prime + f(x(t))x'(t) + g(x(t-\gamma(t))) = e(t).
\]

It is worth stating that the coefficient \( c(t) \) is sign-changing, which is different from the corresponding ones of the previous work.

Gao and Lu in [19] settled the existence and uniqueness of periodic solutions to a class of Liénard type \( p \)-Laplacian equation with a deviating argument of the form

\[
\frac{d}{dt} \varphi_p[(x(t) - Cx(t-\tau))] + f(x(t))x'(t) + \beta(t)g(x(t-\tau(t))) = e(t).
\]

It is significant that the approach used to estimate the prior bounds of periodic solutions is creative compared with the known literature.

Gao et al. in [20] dealt with the periodic solutions of a NFDE system

\[
\frac{d}{dt} \varphi_p[(x(t) - Cx(t-\tau))] + f(x(t))x'(t) + \beta(t)g(x(t-\tau(t))) = e(t).
\]

It is meaningful that the coefficient \( \beta(t) \) ahead of the nonlinear term \( g(x(t-\tau(t))) \) is allowed to change sign in this neutral system.

Gao et al. in [21] treated with the existence of periodic solutions to an \( n \)-dimensional generalized Liénard type NFDE system

\[
\frac{d}{dt} \varphi_p[(x(t) - Cx(t-\tau))] + d \frac{d}{dt} \nabla F(x(t-\tau)) + \nabla G(x(t-\delta(t))) = e(t).
\]

It is worth noting that \( C \) is an \( n \times n \) symmetric matrix of constants in the above gradient system, and the restriction of the \( n \)-dimensional vector valued function \( F \) is \( \lim_{R \rightarrow +\infty} L(R)/R^{n-2} \leq l \), where \( L(R) \) := \( \sup_{1 \leq i,j \leq n, \|x\| \leq R} |\partial^2 F/\partial x_i \partial x_j(x)| \), and \( l \geq 0 \) is a constant.
Liang et al. in [22] studied the periodic solutions of a NFDE system
\[
\frac{d}{dt} \phi_p[x'(t) - c(t)x'(t-r)] = f(x(t))x'(t) + \beta(t)g(x(t-\tau(t))) + e(t). \tag{1.3}
\]
Similar to Eq. (1.2), \(\beta(t)\) in Eq. (1.3) is also sign-changeable, and it is interesting that \(c(t)\) is sign-changeable too in Eq. (1.3), which was infrequent in the known papers. However, although the authors in [22] have achieved some excellent results, the delay term \(\tau(t)\) is not embodied in the results, and the condition imposed on \(c(t)\) is \(|c|_{\infty} = \max_{t \in [0,T]} |c(t)| < 1\).

Motivated by the above papers, and observe that periodic problems on gradient system involving sign-changeable are rarely appeared in the known literature, in this paper we are going to study the periodic solutions for NFDE system (1.1) under the cases \(|c|_{\infty} < \frac{1}{2}\) and \(|u|_{\infty} < |u|_0(|u|_0 - 1)\) with \(|c|_{0} = \min_{t \in [0,T]} |c(t)| > 1\), respectively.

Under the above two restrictions, we give the following main results.

**Theorem.** Assume that the following conditions hold:

\[ H_1 \] There exist constants \(r_1 > 0, r_2 > 0, m \geq 0\) and \(d_1 \geq 0\) such that 
(1) \(r_1 |u|^m \leq |\partial G/\partial u| \leq r_2 |u|^m, \forall |u| > d_1, i = 1, 2, \ldots, N.\)
(2) \(u_i \partial G/\partial u_i > 0, \forall |u| > d_1, i = 1, 2, \ldots, N.\)

There is a constant \(\varepsilon > 0\) such that \(M := D_m[(|\beta^-| + \varepsilon)r_2T/r_1 \int_0^T (\beta^+ + \varepsilon) dt]^{1/m} < 1,\) where
\[
H_2 \quad D_m = \begin{cases} 
2^{1-m/m}, & 0 < m < 1, \\
1, & m \geq 1,
\end{cases}
\beta^+ = \max_i \beta(t), \quad \beta^- = \max_i [-\beta(t), 0].
\]

\[ H_3 \] There are constants \(r_3 > 0, n \geq 0\) and \(d_2 \geq 0\) such that \(|\nabla F(u)| \leq r_3 |u|^n, \forall |u| > d_2.\)

Then system (1.1) has at least one \(T\)-periodic solution if

(I) \(|u|_{\infty} < \frac{1}{2}\) and one of the following conditions holds:
(1) If \(m = n = p - 1\) and \(A_1 + A_2 + A_3 < 1.\)
(2) If \(m = p - 1, n < p - 1\) and \(A_1 + A_3 < 1.\)
(3) If \(m < p - 1, n = p - 1\) and \(A_1 + A_2 < 1.\)
(4) If \(m < p - 1, n < p - 1\) and \(A_1 < 1.\)

or

(II) \(|u|_{\infty} < |u|_0(|u|_0 - 1)\) and one of the following conditions holds:
(1) If \(m = n = p - 1\) and \(A_1^* + A_2^* + A_3^* < 1.\)
(2) If \(m = p - 1, n < p - 1\) and \(A_1^* + A_3^* < 1.\)
(3) If \(m < p - 1, n = p - 1\) and \(A_1^* + A_2^* < 1.\)
(4) If \(m < p - 1, n < p - 1\) and \(A_1^* < 1.\)

where
\[
A_1 = \frac{|c|_{\infty}(1 + |c|_{\infty})^{p-1}}{(1-|c|_{\infty})^p}, \quad A_2 = \frac{C_n r_2 N^{n/2} T^{p-1}}{(1-|c|_{\infty})^p(1-M)^{p-1}}, \quad A_3 = \frac{2^{p-1} r_3 N^{n/2} T^{p-1}}{(1-|c|_{\infty})^p(1-M)^{p-1}},
\]
\[
A_1^* = \frac{|c|_{\infty}(1 + |c|_{\infty})^{p-1}}{(c_0 - 1)^p}, \quad A_2^* = \frac{C_n r_2 N^{n/2} T^{p-1}}{(|c_0 - 1|^p(1-M)^{p-1}}, \quad A_3^* = \frac{2^{p-1} r_3 N^{n/2} T^{p-1}}{(|c_0 - 1|^p(1-M)^{p-1}},
\]
\[
C_n = \begin{cases}
2^{p-1}, & n \geq 1, \\
1, & 0 < n < 1, \quad \theta_1 = \max_{t \in I} \frac{1}{1 - \delta'(\gamma(t))},
\end{cases}
\]
From this theorem, it is significant that our results are related with the delay term $\delta(t)$ and the damping term $\nabla F(u)$ of system (1.1). Moreover, compared with the aforementioned papers, it is clear that [5,6,13,17,20,22] studied with the special cases of the system (1.1).

This paper is organized as follows. In Section 2, we begin by introducing some important notations and lemmas, which this paper is based on. In Section 3, combining the continuation theorem with the results of Section 2, we prove the proposed theorem on the existence of periodic solutions for system (1.1), the results of which generalize and improve on the corresponding ones in [5,6,13,17,20,22]. In the last section, we give explicit examples showing that the assumptions in Section 3 are feasible.

2. Some notations and preliminary results

For simplicity, throughout the paper we will introduce some notations: $|\cdot|$ will denote absolute value and the Euclidean norm on $\mathbb{R}^N$ with $|a| = (\sum_{i=1}^{N} |a_i|^2)^{1/2}$, $\forall a = (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$. For $N \geq 1$, we will set $I = [0, T]$, $C = C(I, \mathbb{R}^N)$, $C^1 = C^1(I, \mathbb{R}^N)$, $C_T = \{ u \in C[u(0) = u(T)] \}$ with the norm $|u|_{\infty} = \max_{t \in I} |u(t)|$, $C_T^1 = \{ u \in C^1[u(0) = u(T), u'(0) = u'(T)] \}$ with the norm $\|u\| = |u|_{\infty}, |u'|_{\infty}$. Obviously, $C_T$ and $C_T^1$ are Banach spaces. On the other hand, we let

$$L : D(L) \subset X \to Y, \quad Lx = x' = \begin{pmatrix} Au' \\ v' \end{pmatrix},$$

$$N : X \to Y, \quad Nx = \begin{pmatrix} -\frac{d}{dt} \nabla F(u(t)) - \beta(t)\nabla G(u(t - \delta)) + c(t) \\ \phi_q(v(t)) \end{pmatrix},$$

and denote the continuous linear operator:

$$A : C_T^1 \to C_T^1, \quad [Au](t) = u(t) - c(t)u(t - \tau).$$

Then we have the following lemmas.

**Lemma 2.1.** If $|c(t)| \neq 1$, then operator $A$ has continuous bounded inverse $A^{-1}$ on $C_T$, which satisfies the following relationships:

$$[A^{-1}f](t) = f(t) + \sum_{j=1}^{\infty} \prod_{i=1}^{j} c[t-(i-1)\tau]f(t-j\tau) \quad \text{for } |c|_{\infty} < 1, \forall f \in C_T,$$

$$[A^{-1}f](t) = -\sum_{j=1}^{\infty} \prod_{i=1}^{j} c(t+i\tau)f(t+j\tau) \quad \text{for } |c|_0 > 1, \forall f \in C_T.$$

(2)

$$\int_0^T |[A^{-1}f](t)|^p \, dt \leq \frac{1}{(1-|c|_\infty)^p} \int_0^T |f(t)|^p \, dt \quad \text{for } |c|_{\infty} < 1, \forall f \in C_T,$$

$$\int_0^T |[A^{-1}f](t)|^p \, dt \leq \frac{1}{(|c|_0-1)^p} \int_0^T |f(t)|^p \, dt \quad \text{for } |c|_0 > 1, \forall f \in C_T.$$

**Proof.** The proof of this lemma, except the second part of conclusion (2), can be found straightforwardly from paper [17,22]. Therefore, here we will give the proof of the second part of conclusion (2) only.
From the second part of conclusion (1), we can find that

$$\|A^{-1}f(t)\| \leq \sum_{j=1}^{\infty} \frac{1}{|c_0^j|} |f(t + j\tau)|$$

Hence, we have

$$\int_0^T \|A^{-1}f(t)\|^p \, dt \leq \int_0^T \|A^{-1}f(t)\|^{p-1} \|A^{-1}f(t)\| \, dt$$

$$\leq \sum_{j=1}^{\infty} \frac{1}{|c_0^j|} \left( \int_0^T \|A^{-1}f(t)\|^p \, dt \right)^{(p-1)/p} \left( \int_0^T |f(t)|^p \, dt \right)^{1/p}$$

i.e.,

$$\left( \int_0^T \|A^{-1}f(t)\|^p \, dt \right)^{1/p} \leq \sum_{j=1}^{\infty} \frac{1}{|c_0^j|} \left( \int_0^T |f(t)|^p \, dt \right)^{1/p},$$

which implies that the second part of conclusion (2) holds. □

**Lemma 2.2** (Mawhin [1]). Suppose that $X$ and $Y$ are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \Omega \to Y$ is $L$-compact in $\overline{\Omega}$. If

1. $Lx \neq \lambda Nx$, $\forall (x, \lambda) \in (D(L) \cap \partial \Omega) \times ]0, 1[$.
2. $N\lambda \notin \text{Im } L$, $\forall x \in \text{Ker } L \cap \partial \Omega$.
3. $\text{deg}[JQN, \Omega \cap \text{Ker } L, 0] \neq 0$, where $J : \text{Im } Q \to \text{Ker } L$ is an isomorphism.

Then the equation $Lx = Nx$ has a solution in $D(L) \cap \overline{\Omega}$.

In order to use Mawhin’s continuation theorem to study the existence of $T$-periodic solutions for system (1.1), we reduce system (1.1) into the following form:

$$\begin{cases}
[Au'](t) = \phi_q(v(t)) = |v(t)|^{q-2}v(t), \\
v'(t) = -\frac{d}{dt}F(u(t-\tau)) - \beta(t)\nabla G(u(t-\delta(t))) + e(t),
\end{cases} \quad (2.3)$$

where $q$ is conjugate to $p$. Obviously, if $x(\cdot) = (u(\cdot), v(\cdot))^T$ is the $T$-periodic solution of Eq. (2.3), then $x(\cdot)$ must be the $T$-periodic solution of Eq. (1.1). Therefore, the problem of finding a $T$-periodic solution for Eq. (1.1) reduces to finding one for Eq. (2.3).

From (2.1) and (2.2), it is easily verified that Eq. (2.3) can be converted to the operator equation $Lx = Nx$. Moreover, from the definition of $L$, $\text{Im } L = \{u : u \in C^1_T, \int_0^T u(s) \, ds = 0\}$, $\text{Ker } L = \mathbb{R}^N$. So $L$ is a Fredholm operator with index zero.

Define

$$Pu = \frac{1}{T} \int_0^T u(s) \, ds, \quad Qy = \frac{1}{T} \int_0^T v(s) \, ds.$$


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Clearly, \( \text{Ker } L = \text{Im } Q = \mathbb{R}^N \) and
\[
[K_N](t) = \int_0^T G(t,s)v(s) \, ds,
\]
where
\[
G(t,s) = \begin{cases} 
\frac{s}{T}, & 0 \leq s < t \leq T, \\
\frac{s-T}{T}, & 0 \leq t \leq s \leq T.
\end{cases}
\]

From (2.2) and (2.4), it is easy to find that \( N \) is \( L \)-compact on \( \Omega \), where \( \Omega \) is an arbitrary open bounded subset of \( CT \).

3. Proof of main results

In this section, through the previous lemmas, we prove the theorem presented in the Introduction.

**Proof.** Consider the operator equation \( Lx = \lambda Nx, \lambda \in ]0,1] \). This implies that
\[
\frac{d}{dt} \phi_p[x'(t)-c(t)x'(t-\tau)] + \lambda \frac{d}{dt} \nabla F(x(t-\tau)) + \lambda \beta(t) \nabla G(x(t-\delta(t))) = \lambda e(t), \quad \lambda \in ]0,1].
\]

(3.1)

Let \( \Omega_1 = \{ x \in CT : Lx = \lambda Nx, \lambda \in ]0,1] \} \). If \( x(\cdot) = (u(\cdot),v(\cdot))^T \in \Omega_1 \), then Eq. (3.1) is equivalent to the following systems:
\[
\begin{cases} 
[Au'](t) = \lambda \phi_q(v(t)) = \lambda |v(t)|^{q-2}v(t), \\
v'(t) = -\lambda \frac{d}{dt} \nabla F(u(t-\tau)) - \lambda \beta(t) \nabla G(u(t-\delta(t))) + \lambda e(t).
\end{cases}
\]

(3.2)

It follows from the first equation of Eq. (3.2) that \( v(t) = \phi_p[(1/\lambda)(Au')](t) \), we combine this with the second equation of Eq. (3.2) and obtain
\[
\frac{d}{dt} \phi_p[(Au')(t)] + \lambda \frac{d}{dt} \nabla F(u(t-\tau)) + \lambda \beta(t) \nabla G(u(t-\delta(t))) = \lambda \beta e(t), \quad \lambda \in ]0,1].
\]

(3.3)

Integrating both sides of Eq. (3.3) on the interval \([0,T]\), we have
\[
\int_0^T \beta(t) \nabla G(u(t-\delta(t))) \, dt = 0,
\]
i.e.,
\[
\int_0^T \beta(t) \frac{\partial}{\partial u_i} G(u(t-\delta(t))) \, dt = 0, \quad i = 1, 2, \ldots, N.
\]

(3.4)

Setting \( \beta^+ = \max_{i \in I} \beta(t), 0 \), \( \beta^- = \max_{i \in I} (-\beta(t), 0) \), then it is easily verified that \( \beta^+ \) and \( \beta^- \) are \( T \)-periodic and \( \beta(t) = \beta^+ - \beta^- \). Therefore, from Eq. (3.4) and using integral mean
value theorem, we see that there is a constant \( \xi \in [0, T] \) such that

\[
\frac{\partial}{\partial u_i} G(u(\xi-\delta(\xi))) \int_0^T (\beta^+ + \varepsilon) \, dt = \int_0^T (\beta^- + \varepsilon) \frac{\partial}{\partial u_i} G(u(t-\delta(t))) \, dt.
\]

(3.5)

Now, we claim that

\[
|u_i(u(\xi-\delta(\xi)))| \leq M|u_i|_\infty + N^*, \quad i = 1, 2, \ldots, N,
\]

(3.6)

where \( M \) defined in \([H_2]\), \( N^* = D_m((|\beta^-|_\infty + \varepsilon)G_{d_1}T/r_1 \int_0^T (\beta^+ + \varepsilon) \, dt)^{1/m} + d_1 \), and \( G_{d_1} = \max_{|u_i| \leq d_1} |\partial G/\partial u_i| \).

Case (i) \( |u_i(u(\xi-\delta(\xi)))| \leq d_1 \): It is easily verified that Eq. (3.6) holds clearly.

Case (ii) \( |u_i(u(\xi-\delta(\xi)))| > d_1 \): Denote

\[
E_1 = \{ t : t \in [0, T], |u_i(t-\delta(t))| > d_1 \}, \quad E_2 = \{ t : t \in [0, T], |u_i(t-\delta(t))| \leq d_1 \}.
\]

Then, it follows from Eq. (3.5) and \([H_1](1)\) that

\[
r_1|u_i(\xi-\delta(\xi))|^m \int_0^T (\beta^+ + \varepsilon) \, dt \leq (|\beta^-|_\infty + \varepsilon) \left( \int_{E_1} + \int_{E_2} \right) \int_0^T \left| \frac{\partial}{\partial u_i} G(u(t-\delta(t))) \right| \, dt
\leq (|\beta^-|_\infty + \varepsilon)r_2 T|u_i|_\infty + (|\beta^-|_\infty + \varepsilon)G_{d_1}T.
\]

Therefore, we have

\[
|u_i(u(\xi-\delta(\xi)))| \leq D_m \left[ (|\beta^-|_\infty + \varepsilon)r_2 T \right]^{1/m} \left[ \frac{1}{r_1} \int_0^T (\beta^+ + \varepsilon) \, dt \right] |u_i|_\infty + D_m \left[ (|\beta^-|_\infty + \varepsilon)G_{d_1}T \right]^{1/m} \left[ \frac{1}{r_1} \int_0^T (\beta^+ + \varepsilon) \, dt \right].
\]

This proves Eq. (3.6).

Let \( \xi-\delta(\xi) = k^* T + \bar{\xi} \), where \( k^* \) is an integer and \( \bar{\xi} \in [0, T] \), we combine this with Eq. (3.6) and obtain

\[
|u_i(t)| \leq M|u_i|_\infty + N^* + \int_{\bar{\xi}}^t |u'(s)| \, ds, \quad t \in [\bar{\xi}, \bar{\xi} + T]
\]

and

\[
|u_i(t)| = |u_i(t-T)| \leq M|u_i|_\infty + N^* + \int_{t-T}^{\bar{\xi}} |u'(s)| \, ds, \quad t \in [\bar{\xi}, \bar{\xi} + T].
\]

Combining the above two inequalities, we find that

\[
|u_i|_\infty \leq M|u_i|_\infty + N^* + \frac{1}{2} \int_0^T |u'_i(t)| \, dt, \quad i = 1, 2, \ldots, N.
\]

(3.7)

In terms of \([H_2]\), then from Eq. (3.7), we have

\[
|u_i|_\infty \leq \frac{N^*}{1-M} + \frac{1}{1-M} \int_0^T |u'_i(t)| \, dt, \quad i = 1, 2, \ldots, N.
\]

(3.8)

i.e.,

\[
|u|_\infty \leq \frac{N^{1/2}N^*}{1-M} + \frac{N^{1/2}}{1-M} \int_0^T |u'(t)| \, dt.
\]

(3.8*)
On the other hand, multiplying both sides of Eq. (3.3) by \( u(t) \) and integrating over \([0, T]\), we get

\[
\int_0^T u(t) \, d\phi_p[(Au')(t)] + \lambda^p \int_0^T u(t) \, d\nabla F(u(t-\tau)) + \lambda^p \int_0^T \beta(t) \nabla G(u(t-\delta(t)))u(t) \, dt = \lambda^p \int_0^T e(t)u(t) \, dt.
\]  

(3.9)

In view of

\[
\int_0^T u(t) \, d\phi_p[(Au')(t)] = - \int_0^T (Au')(t) \, dt - \int_0^T c(t)u'(t-\tau)\phi_p[(Au')(t)] \, dt.
\]

Then Eq. (3.9) can be rewritten into the following form:

\[
\int_0^T (Au')(t) \, dt = - \int_0^T c(t)u'(t-\tau)\phi_p[(Au')(t)] \, dt - \lambda^p \int_0^T \nabla F(u(t-\tau))u'(t) \, dt + \lambda^p \int_0^T \beta(t) \nabla G(u(t-\delta(t)))u(t) \, dt - \lambda^p \int_0^T e(t)u(t) \, dt.
\]

By using Hölder’s inequality and Minkowski inequality, we have

\[
\int_0^T (Au')(t) \, dt \leq |c|_\infty \left( \int_0^T |u'(t-\tau)| \, dt \right)^{1/p} \left[ \left( \int_0^T |u'(t-\tau)|^{p/q} \, dt \right)^{p/q} \right]^{1/p} + \int_0^T |\nabla F(u(t-\tau))|u'(t) \, dt + \int_0^T |\beta(t) \nabla G(u(t-\delta(t)))|u(t) \, dt + |e|_\infty \int_0^T |u(t)| \, dt
\]

\[
\leq |c|_\infty \left( \int_0^T |u'(t)| \, dt \right)^{1/p} \left[ 1 + |c|_\infty^{p/q} \left( \int_0^T |u'(t)| \, dt \right)^{1/q} \right] + \int_0^T |\nabla F(u(t-\tau))|u'(t) \, dt + |\beta|_\infty \int_0^T |\nabla G(u(t-\delta(t)))|u(t) \, dt + |e|_\infty \int_0^T |u(t)| \, dt
\]

\[
\leq |c|_\infty (1 + |c|_\infty)^{p-1} \int_0^T |u'(t)| \, dt + \int_0^T |\nabla F(u(t-\tau))|u'(t) \, dt + |\beta|_\infty \int_0^T |\nabla G(u(t-\delta(t)))|u(t) \, dt + |e|_\infty T|u|_\infty.
\]  

(3.10)

Moreover, from \([H_3]\) and (3.8*), we obtain

\[
\lambda^p \int_0^T |\nabla F(u(t-\tau))|u'(t) \, dt \leq r_3 |u|_0^n \int_0^T |u'(t)| \, dt + F_{d_2} \int_0^T |u'(t)| \, dt
\]

\[
\leq \frac{C_n r_3 N^{n/2}}{(1-M)^n} \left( \int_0^T |u'(t)| \, dt \right)^{n+1} + 1 + \frac{C_n r_3 N^{n/2}}{(1-M)^n} + F_{d_2} \int_0^T |u'(t)| \, dt,
\]

(3.11)

where \( F_{d_2} = \max_{|u|_0 \leq d_2} |\nabla F(u)|. \)

Furthermore, as \( \delta'(t) < 1, \forall t \in \mathbb{R} \), then it is easily seen that the function \( t-\delta(t) \) has a unique inverse denoted by \( \gamma(t) \). Let \( t-\delta(t) = s \), then \( t = \gamma(s) \). Hence, from \([H_1]\)(1), we get

\[
\int_0^T |\nabla G(u(t-\delta(t)))|u(t) \, dt \leq N^{1/2} |u|_0 r_2 \int_{(T-\delta(T)}^{\gamma(s)} \frac{|u(s)|^m}{1-\delta'(\gamma(s))} \, ds + N^{1/2} |u|_\infty G_{d_1} t
\]
\[
\leq N^{1/2} r_1 T |u_i|_\infty^{m+1} + N^{1/2} G_d T |u_i|_\infty, \quad i = 1, 2, \ldots, N.
\]

According to Eq. (3.8), we have
\[
\int_0^T |\nabla G(u(t-\delta(t)))| |u(t)| \, dt \leq \frac{2^{m} N^{1/2} r_1 T}{(1-M)^{m+1}} \left( \int_0^T |u'(t)| \, dt \right)^{m+1} + \frac{N^{1/2} G_d T}{1-M} \int_0^T |u'(t)| \, dt + \frac{2^{m} N^{1/2} r_2 T N^{*m+1}}{(1-M)^{m+1}} + \frac{N^{1/2} G_d T N^*}{1-M}.
\]

Substituting Eqs. (3.11) and (3.12) into Eq. (3.10), we obtain
\[
\int_0^T |(Au'(t))|^p \, dt \leq |c|_\infty (1 + |c|_\infty)^{p-1} \int_0^T |u'(t)|^p \, dt + \frac{C_n r_3 N^{n/2}}{(1-M)^n} \left( \int_0^T |u'(t)|^p \, dt \right)^{n+1} + \frac{2^{m} N^{1/2} r_1 T}{(1-M)^{m+1}} \left( \int_0^T |u'(t)|^p \, dt \right)^{m+1} + \theta_2 \int_0^T |u'(t)| \, dt + \theta_3 \leq |c|_\infty (1 + |c|_\infty)^{p-1} \int_0^T |u'(t)|^p \, dt + \frac{C_n r_3 N^{n/2} T^{(p-1)(n+1)/p}}{(1-M)^{n+1}} \left( \int_0^T |u'(t)|^p \, dt \right)^{(m+1)/p} + \theta_2 \int_0^T |u'(t)| \, dt + \theta_3,
\]

where
\[
\theta_2 = \frac{C_n r_3 N^{n+1} N^{n/2}}{(1-M)^n} + \frac{G_d T N^{1/2} |\beta|_\infty}{1-M} + \frac{|c|_\infty T N^{1/2}}{1-M} + F_{d_2},
\]
\[
\theta_3 = \frac{2^{m} N^{1/2} r_2 T N^{*m+1}}{(1-M)^{m+1}} \frac{N^{1/2} |\beta|_\infty}{1-M} + \frac{G_d T N^* N^{1/2} |\beta|_\infty}{1-M} + \frac{|c|_\infty T N^* N^{1/2}}{1-M}.
\]

From Lemma 2.1(2), we get
\[
\begin{align*}
\int_0^T |u'(t)|^p \, dt &\leq \frac{1}{(1-|c|_\infty)^p} \int_0^T |(Au'(t))|^p \, dt \quad \text{for } |c|_\infty < 1, \\
\int_0^T |u'(t)|^p \, dt &\leq \frac{1}{(|c|_0 - 1)^p} \int_0^T |(Au'(t))|^p \, dt \quad \text{for } |c|_0 > 1.
\end{align*}
\]

We combine this with Eq. (3.13) and obtain
\[
\int_0^T |u'(t)|^p \, dt \leq \frac{|c|_\infty (1 + |c|_\infty)^{p-1}}{(1-|c|_\infty)^p} \int_0^T |u'(t)|^p \, dt + \frac{C_n r_3 N^{n/2} T^{(p-1)(n+1)/p}}{(1-|c|_\infty)^p (1-M)^n} \left( \int_0^T |u'(t)|^p \, dt \right)^{(n+1)/p} + \frac{2^{m} N^{1/2} r_1 T^{1+(p-1)(n+1)/p}}{(1-M)^{m+1}} \left( \int_0^T |u'(t)|^p \, dt \right)^{(m+1)/p} + \theta_2 \frac{T^{(p-1)/p}}{(1-|c|_\infty)^p} \left( \int_0^T |u'(t)|^p \, dt \right)^{1/p} + \theta_3 \frac{T^{(p-1)/p}}{(1-|c|_\infty)^p}, \quad |c|_\infty < 1.
\]

\[
(3.14)
\]
and
\[
\int_0^T |u'(t)|^p \, dt \leq \frac{|c|_\infty (1 + |c|_\infty)^{p-1}}{(|c|_0 - 1)^p} \int_0^T |u'(t)|^p \, dt + \frac{C_n T^{n/2} T^{(n+1)/(n+1)}/p}{(|c|_0 - 1)^p (1-M)^p} \left( \int_0^T |u'(t)|^p \, dt \right)^{(n+1)/p} \\
+ \frac{2^m T^{(n+1)/(n+1)}/p}{(|c|_0 - 1)^p (1-M)^{m+1}} \left( \int_0^T |u'(t)|^p \, dt \right)^{(m+1)/p} + \frac{\theta_2 T^{(n-1)/p}}{(|c|_0 - 1)^p} \left( \int_0^T |u'(t)|^p \, dt \right)^{1/p} \\
+ \frac{\theta_1}{(|c|_0 - 1)^p}, \quad |c|_0 > 1.
\] (3.14*)

Next, we will show that \( \int_0^T |u'(t)|^p \, dt \) is bounded.

\textbf{Case 1}: If \( m = n = p - 1 \), in terms of
\[
\begin{cases}
A_1 + A_2 + A_3 < 1 & \text{for } |c|_\infty < 1, \\
A_1^* + A_2^* + A_3^* < 1 & \text{for } |c|_0 > 1,
\end{cases}
\]
and \( 1/p < 1 \), then it follows from Eqs. (3.14) and (3.14*) that \( \int_0^T |u'(t)|^p \, dt \) is bounded obviously.

\textbf{Case 2}: If \( m = p - 1, n < p - 1 \), in view of
\[
\begin{cases}
A_1 + A_3 < 1 & \text{for } |c|_\infty < 1, \\
A_1^* + A_3^* < 1 & \text{for } |c|_0 > 1,
\end{cases}
\]
and \( (n+1)/p < 1 \) and \( 1/p < 1 \), then it follows from Eqs. (3.14) and (3.14*) that \( \int_0^T |u'(t)|^p \, dt \) is bounded clearly.

\textbf{Case 3}: If \( m < p - 1, n = p - 1 \), in view of
\[
\begin{cases}
A_1 + A_2 < 1 & \text{for } |c|_\infty < 1, \\
A_1^* + A_2^* < 1 & \text{for } |c|_0 > 1,
\end{cases}
\]
and \( (m+1)/p < 1 \) and \( 1/p < 1 \), then it follows from Eqs. (3.14) and (3.14*) that \( \int_0^T |u'(t)|^p \, dt \) is also bounded.

\textbf{Case 4}: If \( m = n = p - 1 \), in view of
\[
\begin{cases}
A_1 < 1 & \text{for } |c|_\infty < 1, \\
A_1^* < 1 & \text{for } |c|_0 > 1,
\end{cases}
\]
and \( (n+1)/p < 1, (m+1)/p < 1, \) and \( 1/p < 1 \), then it follows from Eqs. (3.14) and (3.14*) that \( \int_0^T |u'(t)|^p \, dt \) is bounded too.

Combining the above four cases, we can easily see that there is a constant \( M_1 \) such that
\[
\int_0^T |u'(t)|^p \, dt \leq M_1,
\] (3.15)
which together with (3.8*), yields
\[
|u|_\infty \leq \frac{N^{1/2} N^*}{1-M} + \frac{N^{1/2} T^{(p-1)/p} M_1^{1/p}}{1-M} \Delta M_2.
\] (3.16)
Then, from Eq. (3.3), we have
\[
\left| \frac{d}{dt} \phi_p \left[ \frac{1}{\lambda} (Au')(t) \right] \right|_\infty \leq F_{M_2} + |\beta|_\infty G_{M_2} + |c|_\infty \triangleq M_3,
\]
(3.17)
where \( F_{M_2} = \max_{|u| \leq M_2} |(d/dt)NF(u)|, \ G_{M_2} = \max_{|u| \leq M_2} |\nabla G(u)|. \)
Since \( v(t) = \phi_p \left[ (1/\lambda)(Au')(t) \right], \) which together with Eq. (3.17) gives
\[
|v'|_\infty \leq M_3.
\]
(3.18)
Note that \( \int_0^T [A^{-1} \phi_q(v)](t) \, dt = 0, \) so there exists a constant \( \eta \in [0, T] \) such that \( [A^{-1} \phi_q(v)](\eta) = 0, \) we combine this with Lemma 2.1(1) and obtain
\[
\begin{align*}
[A^{-1} \phi_q(v)](\eta) &= \phi_q(v(\eta)) + \sum_{i=1}^{\infty} c[\eta-(i-1)\tau] \phi_q(v(\eta-i\tau)) = 0 \quad \text{for } |c|_\infty < 1, \\
[A^{-1} \phi_q(v)](\eta) &= -\frac{\phi_q(v(\eta+\tau))}{c(\eta+\tau)} \cdot \sum_{i=1}^{\infty} \frac{1}{c(\eta+i\tau)} \phi_q(v(\eta+i\tau)) = 0 \quad \text{for } |c|_0 > 1.
\end{align*}
\]
These imply that
\[
\begin{align*}
|v(\eta)| &= |\phi_q(v(\eta))|^{p-1} \leq \left( \sum_{j=1}^{\infty} |c|_j|\eta|^{j-1} \right)^{p-1} \leq \left( \frac{|c|_\infty}{1-|c|_\infty} \right)^{p-1} |v|_\infty \quad \text{for } |c|_\infty < 1, \\
|v(\eta+\tau)| &= |\phi_q(v(\eta+\tau))|^{p-1} \leq \left( \sum_{j=1}^{\infty} \frac{1}{|c|_j|\eta|^{j-1}} \right)^{p-1} \leq \left( \frac{|c|_0}{|c|_0(|c|_0-1)} \right)^{p-1} |v|_\infty \quad \text{for } |c|_0 > 1.
\end{align*}
\]
(3.19)
It follows from Eqs. (3.18) and (3.19) that
\[
\begin{align*}
|v(t)| &\leq |v(\eta)| + \frac{1}{2} \int_0^T |v'(t)| \, dt \leq \left( \frac{|c|_\infty}{1-|c|_\infty} \right)^{p-1} |v|_\infty + \frac{M_3 T}{2} \quad \text{for } |c|_\infty < 1, \\
|v(t)| &\leq |v(\eta+\tau)| + \frac{1}{2} \int_0^T |v'(t)| \, dt \leq \left( \frac{|c|_\infty}{|c|_0(|c|_0-1)} \right)^{p-1} |v|_\infty + \frac{M_3 T}{2} \quad \text{for } |c|_0 > 1.
\end{align*}
\]
(3.20)
For the first inequality of Eq. (3.20), since \( |c|_\infty < \frac{1}{2}, \) i.e., \( |c|_\infty/(1-|c|_\infty) < 1, \) we can easily see that \( |v|_\infty \) is bounded. Similarly, from the second inequality of Eq. (3.20), as \( |c|_\infty/|c|_0(|c|_0-1)| < 1, |c|_0 > 1, \) we can also see that \( |v|_\infty \) is bounded. So, there is a constant \( M_4 > 0 \) such that
\[
|v|_\infty \leq M_4.
\]
(3.21)
Let \( \Omega_2 = \{ x : x \in \text{Ker } L, Nx = \text{Im } L \}. \) If \( x \in \Omega_2, \) then \( x \in \text{Ker } L \) and \( QNx = 0. \) Clearly, \( |v|^{q-2}v = 0, \) then the second equation of Eq. (3.2) can be reduced into the form \( \int_0^T \beta(t)\nabla G(u) \, dt = 0, \) which together with \( [H_1](2) \) implies that \( |u| \leq d_1 \) and \( \Omega_2 \subset \Omega_1. \)
Now, we let \( \Omega = \{ x : x = (u, v)^T \in \text{C}_T, |u|_\infty < M_2 + 1, |v|_\infty < M_4 + 1 \} \), then \( \Omega_1 \cup \Omega_2 \subset \Omega. \) In terms of Eqs. (3.16) and (3.21), it is easily verified that the first two conditions of Lemma 2.2 are satisfied.
Next, we claim that the last condition of Lemma 2.2 is also satisfied. To this end, we define the isomorphism
\[
J : \text{Im } Q \rightarrow \text{Ker } L, \quad J(u, v) = (v, -u)
\]
and let 
\[ H(x, \mu) = \mu x + (1-\mu)JQN x, \quad (x, \mu) \in \Omega \times [0, 1]. \]
Then, for \((x, \mu) \in \partial(\Omega \cap \text{Ker} L) \times [0, 1]\), we have
\[ x^T H(x, \mu) = \mu(u^2 + v^2) + (1-\mu)\left[ \sum_{i=1}^{N} u_i \partial G \int_{0}^{T} \beta(t) \, dt + |v|^q \right] > 0. \]
Thus,
\[ \text{deg}(JQN, \Omega \cap \text{Ker} L, 0) = \text{deg}(H(x, 0), \Omega \cap \text{Ker} L, 0) = \text{deg}(H(x, 1), \Omega \cap \text{Ker} L, 0) = \text{deg}(I, \Omega \cap \text{Ker} L, 0) \neq 0. \]
Therefore, the last condition of Lemma 2.2 is also satisfied.
According to Lemma 2.2, the operator equation \(Lx = Nx\) has a solution \(x(t) = (u(t), v(t))^T\) on \(\overline{\Omega}\), i.e., system (1.1) has at least one periodic solution \(u(t)\) with \(|u|_\infty \leq M_2\). This completes the proof of the theorem. \(\square\)

**Remark 1.** If \(\int_{0}^{T} e(t) \, dt \neq 0\), let \(\overline{e}(t) = e(t) - (1/T) \int_{0}^{T} e(s) \, ds\), then \(\int_{0}^{T} \overline{e}(t) \, dt = 0\), and system (1.1) is equivalent to the system
\[ \frac{d}{dt} \phi_p[x'(t)-c(t)x'(t-\tau)] + \frac{d}{dt} \nabla F(x(t-\tau)) + \beta(t)\nabla G(x(t-\delta(t)) - \frac{1}{T} \int_{0}^{T} e(t) \, dt = \overline{e}(t). \] (3.22)
Thus, we can study system (3.22) with the conclusions in the theorem. \(\square\)

If \(F(x(t-\tau)) \equiv F(x(t))\), then system (1.1) can be reduced to the form
\[ \frac{d}{dt} \phi_p[x'(t)-c(t)x'(t-\tau)] + \frac{d}{dt} \nabla F(x(t)) + \beta(t)\nabla G(x(t-\delta(t))) = e(t). \] (3.23)
Then we have the following corollary.

**Corollary.** Suppose that \([H_1]-[H_2]\) in the theorem hold, then system (3.23) has at least one \(T\)-periodic solution if

(I) \(|u|_\infty < \frac{1}{2}\) and one of the following conditions holds:
1. If \(m = p-1\) and \(\Delta_1 + \Delta_3 < 1\).
2. If \(m < p-1\) and \(\Delta_1 < 1\).

or

(II) \(|u|_\infty < |u_0| (|u_0| - 1)\) and one of the following conditions holds:
1. If \(m = p-1\) and \(\Delta_1^* + \Delta_3^* < 1\).
2. If \(m < p-1\) and \(\Delta_1^* < 1\),

where \(\Delta_1, \Delta_1^*, \Delta_3, \Delta_3^*\) defined as in the theorem. \(\square\)

4. Examples

In this section we apply the general results obtained in the preceding section to some specific examples.
Example 1. Consider system
\[ \frac{d}{dt} \phi_4 \left( x'(t) - \frac{1}{10} \cos 20 \pi t x'(t - \tau) \right) + \frac{d}{dt} \left( \sin \left( \frac{x^3(t - \tau)}{10} \right) + \left( \frac{\sqrt{3}}{40} + \frac{1}{20} \sin 20 \pi t \right) \frac{x^3(t - \frac{1}{2} \sin 20 \pi t)}{10} \right) = \cos 20 \pi t. \]  
(4.1)

Comparing with system (1.1), one has \( N = 1, \ p = 4, \ c(t) = \frac{1}{10} \cos 20 \pi t, \ \nabla F(u) = \frac{1}{10} \sin u^3, \ \beta(t) = \frac{\sqrt{3}}{40} + \frac{1}{20} \sin 20 \pi t, \ \nabla G(v) = \frac{1}{10} v^3, \ \delta(t) = \frac{1}{2} \sin 20 \pi t, \ e(t) = \cos 20 \pi t. \)

Therefore, system (4.1) has at least one \( \frac{1}{10} \)-periodic solution according to the theorem.

Example 2. Consider the following system:
\[ \frac{d}{dt} \phi_4 \left[ x'(t) - c(t) x'(t - \tau) \right] + \frac{d}{dt} \left( \sin \left( \frac{x^3(t - \tau)}{10} \right) + \left( \frac{\sqrt{3}}{40} + \frac{1}{20} \sin 20 \pi t \right) \frac{x^3(t - \frac{1}{2} \sin 20 \pi t)}{10} \right) = \cos 20 \pi t, \]  
(4.2)

where
\[ c(t) = \begin{cases} 
3 + \sin 20 \pi t, & t \in \left[ \frac{k}{10}, \frac{k}{10} + \frac{1}{20} \right], \\
-3 - \sin 20 \pi t, & t \in \left( \frac{k}{10} + \frac{1}{20}, \frac{1}{10} + \frac{1}{20} \right],
\end{cases} \]

\( k \) is an integer, and the rest parameters are the same as in Example 1.

In view of \( |c|_{\infty} = 4, |c|_0 = 3 \), i.e., \( |c|_{\infty} < |c|_0(|c|_0 - 1) \) and
\[ \frac{|c|_{\infty}(1 + |c|_{\infty})^3}{(|c|_0 - 1)^4} + \frac{C_n r_3 T^3}{(|c|_0 - 1)^4(1 - M)^3} + \frac{8r_2 \delta_1 |\beta|_{\infty} T^4}{(|c|_0 - 1)^4(1 - M)^3} \approx 0.0122 < 1. \]

Hence, system (4.2) has at least one \( \frac{1}{10} \)-periodic solution by the theorem.

Remark 2. Obviously, \( c(t) \) and \( \beta(t) \) are sign-changeable according to the above two examples, and the results achieved in [5,6,13,17,20,22] and the reference therein cannot be applied to systems (4.1) and (4.2). Therefore, our results in this paper generalize and improve on the associated results in [5,6,13,17,20,22], and implies that the results in the present paper are essentially new.

Acknowledgements

The authors gratefully acknowledge the support of the National Science Foundation for Distinguished Young Scholars of China (NSFDYSC) through Grant no. 10425209, the National Natural Science Foundation of China (NNSFC) through Grant nos. 10732020, 11072008 and 10972011, the Funding Project for Academic Human Resources Development in Institutions of Higher Learning under the Jurisdiction of Beijing Municipality (PHRIHLB).
References